• Implicit function theorem & Inverse function theorem

Implicit function theorem $F: \Omega (\subseteq \mathbb{R}^{n+k}) \to \mathbb{R}^k$ be C^1.

$(a,b) \in \Omega$ and $F(a,b) = C$. $\underset{\mathbb{R}^n}{\overset{(x,}{}}\;\underset{\mathbb{R}^k}{\overset{y)}{}}$

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ \vdots \\ F_k(x,y) \end{pmatrix}$$

If $\left[\frac{\partial F_i}{\partial y_j}(a,b) \right]_{\substack{1 \le i \le k \\ 1 \le j \le k}}$ is invertible.

$\exists \varphi : \underset{\substack{\uparrow \\ a}}{U}^{\subseteq \mathbb{R}^n} \longrightarrow \underset{\substack{\uparrow \\ b}}{V}^{\subseteq \mathbb{R}^k}$ s.t. $\varphi(a) = b$ and

$F(x, \varphi(x)) = C$ for all $x \in U$. Moreover φ is C^1.

Inverse function thm $f: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1.

$f(a) = b$.

If $Df(a)$ is invertible, then

n×n matrix

$$a \in U (\subseteq \Omega) \underset{\longleftarrow}{\xrightarrow{f}} V (\subseteq \mathbb{R}^n) \ni b \qquad \exists g : V \to U$$

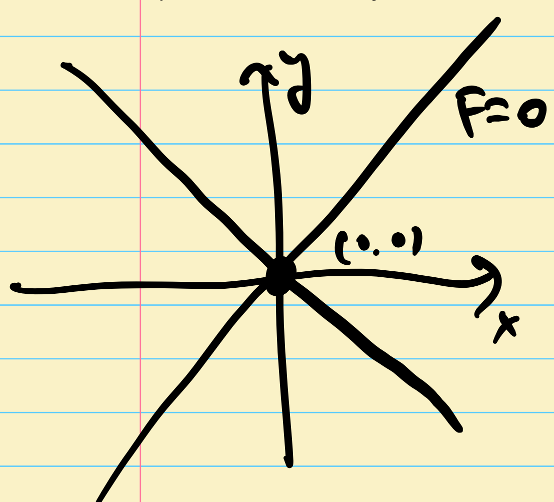
s.t. $g(b)=a$ and $\begin{cases} g(f(x))=x & \forall x\in U \\ f(g(y))=y & \forall y\in V \end{cases}$

Also g is C^1, $Dg(y) = Df(g(y))^{-1}$.

In both theorems, we assume a Jacobian matrix to be invertible. If a Jacobian matrix fails to be ″ , the theorems are inconclusive.

eg ① Implicit function theorem.

1) $F(x,y) = x^2 - y^2 = 0$ $\quad y=y(x)$ near $(0,0)$?

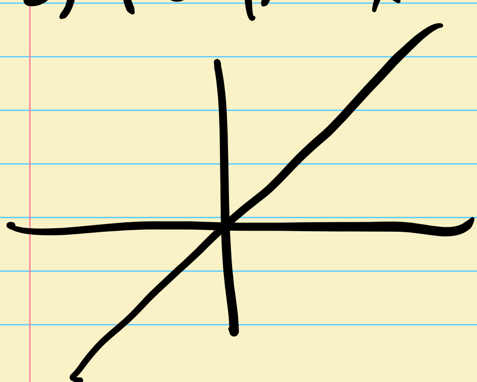


$$\frac{\partial F}{\partial y}(0,0) = -2y\Big|_{(0,0)} = 0$$

Implicit function theorem says nothing.

y is not a function of x near $(0,0)$.

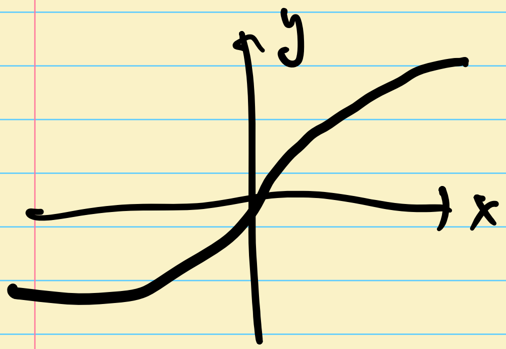
2) $F(x,y) = x^3 - y^3 = 0$ $\quad y=y(x)$ near $(0,0)$?



$\frac{\partial F}{\partial y}(0,0) = 0$ $\quad$ IFT says nothing.

$y=x$ near $(0,0)$. (In fact $y=x$ globally)

3) $F(x,y) = x - y^3 = 0$ $\quad y = y(x)$ near $(0,0)$?



$\frac{\partial f}{\partial y}(0,0) = 0$ IFT says nothing.

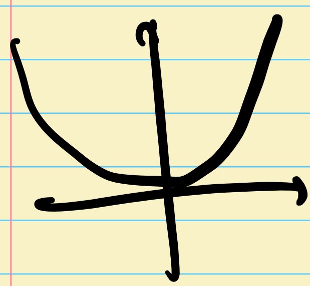
$y = \sqrt[3]{x}$. But note that

$y = \sqrt[3]{x}$ is not differentiable

at $x = 0$.

② Inverse function theorem

1) $f(x) = x^2$ $\quad f'(0) = 0 \Rightarrow$ Inverse function theorem says nothing



Since f is not injective near $x=0$,
no local inverse.

2) $f(x) = x^3$. $\quad f'(0) = 0$

$g(y) = \sqrt[3]{y}$ is an inverse of f near 0.

However $g(y) = \sqrt[3]{y}$ is not differentiable at 0.

Exercise Let $F(x,y,z): \mathbb{R}^3 \to \mathbb{R}$ C^1-function.

$F(x_0, y_0, z_0) = c$.

Suppose $\nabla F = (y, e^x, 1)$

(1) Show that near (x_0, y_0, z_0), z can be expressed by a function on x, y $z = z(x,y)$

(2) Show that for $z(x,y)$ from (1),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

and compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (x_0, y_0, z_0).

(sol) (1) $\frac{\partial F}{\partial z}(x_0, y_0, z_0) = 1 \neq 0$

By Implicit function theorem, near (x_0, y_0, z_0)

z can be expressed as a function of x, y.

(2) $\underbrace{F(x, y, z(x,y))}_{\text{Near}(x_0,y_0,z_0)} = F(x_0, y_0, z_0)$

Take implicit differentiation w.r.t x.

$$F_x + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Similarly, implicit differentiation w.r.t. y $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$$\frac{\partial z}{\partial x}(x_0, y_0, z_0) = -\frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)} = -\frac{y_0}{1} = -y_0$$

$$\frac{\partial z}{\partial y}(x_0, y_0, z_0) = -\frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)} = -\frac{e^{x_0}}{1} = -e^{x_0}.$$

Differentiability of $f: \mathbb{R}^n \to \mathbb{R}$ = existence of linear approximation.

f is differentiable at a if $\exists\, L(x) = f(a) + c_1(x_1 - a_1) \dots + c_n(x_n - a_n)$

$a = (a_1, \dots, a_n)$

s.t. $$\lim_{x \to a} \frac{|f(x) - L(x)|}{\|x - a\|} = 0.$$

In this case, we must have $c_i = \frac{\partial f}{\partial x_i}(a)$

i.e. $(c_1, \dots, c_n) = \nabla f(a)$.

$f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a if

$$\exists\, L(x) = f(a) + \underbrace{M}_{m \times n \text{ matrix}} \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix} \quad \text{s.t.}$$

$$\lim_{x \to a} \frac{\|f(x) - L(x)\|}{\|x - a\|} = 0.$$

In this case, we have $M = Df(a)$

If we write $f = (f_1, \cdots, f_m)$,

f is differentiable $\iff$ $f_1, \cdots, f_m$ are differentiable.

Exercise $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (f_1(x,y), f_2(x,y))$

$f_1(x,y) = x^2 + y^2$

$$f_2(x,y) = \begin{cases} (x^2+y^2) \cdot \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute $Df(0,0)$.

(b) Show that f is differentiable.

(Sol) $$Df(0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 2x & 2y \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(0,0)}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{\partial f_2}{\partial x}(0,0) & \frac{\partial f_2}{\partial y}(0,0) \end{pmatrix}$$

$$\frac{\partial f_2}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{t^2 \cdot \sin\left(\frac{1}{\sqrt{t^2}}\right)}{t}$$

$$= \lim_{t\to 0} t \cdot \sin\left(\frac{1}{|t|}\right) = 0$$

by Sandwich theorem.

Similarly, $\frac{\partial f_2}{\partial y}(0,0) = 0$.

$\therefore Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) We need to check both f_1 and f_2 are differentiable.

Since f_1 is a polynomial, it is differentiable.

To show that f_2 is differentiable, we need to show that $\lim_{(x,y)\to(0,0)} \frac{|f_2(x,y) - L(x,y)|}{\|(x,y)\|} = 0$

$$= \lim_{(x,y)\to(0,0)} \frac{|f_2(x,y)|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}$$

$$= \lim_{r\to 0} \frac{r^2 \sin\left(\frac{1}{r}\right)}{r} = \lim_{r\to 0} r\sin\left(\tfrac{1}{r}\right) = 0$$

By Sandwich theorem.

$\therefore f_2$ is differentiable

$\therefore f$ is differentiable. $\square$

Second derivative test $f: \mathbb{R}^2 \to \mathbb{R}$ is C^2.

a is a critical point of f (i.e. $\nabla f(a) = 0$)

① $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$ at $a \Rightarrow a$ is local min

② " > 0, $f_{xx} < 0$ " $\Rightarrow a$ is local max

③ " < 0 $\Rightarrow$ saddle point

④ $f_{xx}f_{yy} - f_{xy}^2 = 0$ $\Rightarrow$ inconclusive.

Exercise $f(x,y) = x^4 - x^2 + 2xy + y^2$

Find all points in $\mathbb{R}^2$ where local minimum of f occur.

(S.1) $\nabla f = (4x^3 - 2x + 2y,\ 2x + 2y)$

$\nabla f = (0, 0)$ if $\begin{cases} 4x^3 - 2x + 2y = 0 \\ 2x + 2y = 0 \end{cases} \Rightarrow (x, y) = (0,0),\ (1,-1),\ (-1,1)$

$$Hf = \begin{pmatrix} 12x^2 - 2 & 2 \\ 2 & 2 \end{pmatrix}$$

	$(0,0)$	$(1,-1)$	$(-1,1)$
Hf	$\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$
$\det Hf$	-8	16	16
$f_{xx}(0,0)$	-2	10	10

By second derivative test, f has local minimum at $(-1,1)$ and $(1,-1)$.

Taylor's theorem Suppose $f: \mathbb{R}^n \to \mathbb{R}$ C^k function.

Then for $x, a \in \mathbb{R}^n$

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$\vdots$$

$$+ \sum_{i_1, \dots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k})$$

$$+ \varepsilon_k(x, a)$$

with
$$\lim_{x \to a} \frac{\varepsilon_k(x, a)}{\|x - a\|^k} = 0.$$

Exercise Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a C^3-function. Suppose

$$Df(0,0) = \vec{0}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f_{xxx}(0,0) = f_{yyy}(0,0) = f_{xyy}(0,0) = 0$$

$$f_{xxy}(0,0) > 0$$

Is it possible for f to have a local minimum at $(0,0)$? If yes, give an example; if no, explain why.

(Sol) No. It is not possible.

By Taylor's theorem,

$$f(x,y) = f(0,0) + \nabla f(0,0)\cdot(x,y) + \frac{1}{2}(x,y)\,Hf(0,0)\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{3!}\Big(f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3\Big) + \varepsilon(x,y)$$

Since f is C^3, by mixed derivative theorem, higher derivatives up to order 3 can be changed.

$$= f(0,0) + 0 + 0 + \frac{1}{6}\cdot 3 f_{xxy}(0,0)x^2y + \varepsilon(x,y)$$

$$= f(0,0) + \frac{1}{2} f_{xxy}(0,0)\, x^2y + \varepsilon(x,y)$$

where $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|(x,y)\|^3} = 0$

$C = \frac{1}{2} f_{xxy}(0,0) > 0$

idea. f is similar to $f(0,0) + C \cdot x^2 y$ near $(0,0)$

Cx^2y has no local minimum at $(0,0)$

because if $y > 0 \Rightarrow Cx^2y > 0$

$y < 0 \Rightarrow Cx^2y < 0$

Suppose f has a local minimum at $(0,0)$.

Then $\exists \delta > 0$ s.t. $f(x,y) > f(0,0)$

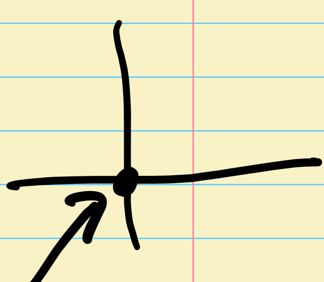
for all $\|(x,y)\| < \delta$.

$$f(x,y) - f(0,0) > 0 \Rightarrow \frac{1}{2} f_{xxy}(0,0)\, x^2 y + \varepsilon(x,y) > 0$$

$$\Rightarrow \varepsilon(x,y) > -\frac{1}{2} f_{xxy}(0,0)\, x^2 y.$$

Consider a curve C parametrized by

$$(x(t), y(t)) = \left(-\frac{\sqrt{2}}{2} t, -\frac{\sqrt{2}}{2} t\right) \quad 0 < t \leq \delta$$



On C,

$$\varepsilon(x(t), y(t)) > -\frac{1}{2} f_{xxy}(0,0) \cdot \left(-\frac{\sqrt{2}}{2} t\right)^2 \left(-\frac{\sqrt{2}}{2} t\right)$$

$$= \frac{1}{4\sqrt{2}} f_{xxy}(0,0)\, t^3$$

Hence $\dfrac{|\varepsilon(x(t), y(t))|}{\|(x(t), y(t))\|^3} = \dfrac{|\varepsilon(x(t), y(t))|}{t^3}$

$$> \frac{1}{4\sqrt{2}} f_{xxy}(0,0)$$

$$> 0$$

This contradicts to $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|\varepsilon(x,y)|}{\|(x,y)\|^3} = 0$.

$\therefore$ f cannot have a local minimum at $(0,0)$.